ON THE STABILITY OF THE TRAJECTORY OF MARKOV PROCESSES

(OB USTOICHIVOSTI TRAEKTORII MARKOVSKIKH PROTSESSOV)

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In [1] criteria for the stability of the solution $X(t) \equiv 0$ of a system $dx/dt = f(X, t, Y(t, \omega))$, where $f(0, t, y) \equiv 0$, and $Y(t, \omega)$ is a Markov random process were established. Here a study is made of the problem of stability of the trajectory of a Markov process with a different definition of stability and under different assumptions relative to the process. Only continuous Markov processes of the diffusion type are considered for which the coefficients of diffusion and transport become zero when $x = 0$. A necessary and sufficient condition for the stability of such processes is found, which is analogous to the fundamental theorem of Liapunov's second method. For the verification of this condition it is sufficient to know the coefficients of diffusion and transport within an arbitrarily small neighborhood of the point $x = 0$. The relation between the stability of a system of ordinary equations, and the stability of stochastic systems obtained from the former by the addition of diffusion is also investigated in this work. It is shown, for example, that in the case when the number of equations in the system $n > 2$, a sufficiently large diffusion will reduce the stability; in the case that $n \leqslant 2$, the stability (the asymptotic one) will be preserved.

1. Formulation of the problem. The object of study in this work will be a random Markov process which can be described [2, p.247] by a stochastic differential equation in vector form

$$\frac{dX}{dt} = b(X(t, \omega), t) + \sigma(X(t, \omega), t)\, \dot{\xi}(t, \omega) \tag{1.1}$$

$$(X(t, \omega) = \{X_1(t, \omega), \ldots, X_n(t, \omega)\}, \qquad b(x, t) = \{b_1(x, t), \ldots, b_n(x, t)\})$$

where $\sigma(x, t)$ is an $n \times n$ matrix, $\{\omega\} = \Omega$ is a set of elementary events, and $\dot{\xi}(t, \omega)$ is n-dimensional "white noise".

In order that the equation (1.1) may have a solution $X(t\ \omega) \equiv 0$, it is necessary that $b(0,\ t) \equiv 0$, and $\sigma(0,\ t) \equiv ((0))$. Besides, just as in the case when $\sigma(x,\ t) \equiv ((0))$, it is necessary to impose conditions on the coefficients under which $X(t,\ \omega) \equiv 0$ will be the unique trajectory, with probability one, which passes through the point $x = 0$. The conditions are given below.

A solution of the equation (1.1) is, as is known, the Markov random process in the n-dimensional Euclidean space E_n with continuous trajectories $X(t,\ \omega)$. In the notation of [3] this Markov process can be denoted by

$$X = \{X\ (t,\omega),\ P_{s,x}\}$$

Here $P_{s,x}(A)$ denotes the probability of the event A under the condition that $X(s,\ \omega) = x$. We shall omit the argument ω in the sequel.

Definition. A trajectory $X(t) \equiv 0$ of the process X is said to be stable for $t \geqslant t_0$, if for every $\varepsilon > 0$, and $s \geqslant t_0$

$$\lim_{x\to 0} P_{s,x}\left\{\sup_{t>s} |X\ (t)| > \varepsilon\right\} = 0 \tag{1.2}$$

In other words, the trajectory $X(t) \equiv 0$ is stable if the probability, that $X(t)$ leaves an ε-neighborhood of the point $x = 0$ even once, can be made arbitrarily small if the position of the trajectory at the initial instant of time s is chosen near enough to $x = 0$.

The following differential operator is related to the process X

$$L\,(t,x) \equiv \sum_{i,j=1}^{n} a_{ij}\,(t,x)\,\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i\,(t,x)\,\frac{\partial}{\partial x_i} \qquad (\|\,a_{ij}\,(t,x)\,\| = \sigma\sigma^*)$$

Here σ^* is a matrix which is the transpose of σ.

For the establishment of the connection between the theorems given below and the theorems of Liapunov, it is well to keep in mind that the expression $Lu(t,\ x)$, for any twice continuously differentiable function $u(t,\ x)$, can be considered as the mean value of the derivative of the function $u(t,\ x)$ along the trajectory of the Markov process X which emerges from the point x at the instant of time t.

From what has been said, it follows that in the case under consideration

$$b_i\ (t,\ 0) \equiv 0, \qquad a_{ij}\ (t,\ 0) \equiv 0 \tag{1.3}$$

In the sequel, unless stated otherwise, we shall study a homogeneous, in time, process for which $L(t, x) \equiv L(x)$; hereby $P_{s,x}(\cdot) \equiv P_x(\cdot)$.

Let us introduce the notation

$$|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = r$$

It is necessary to make the statement of the problem more precise since the equation (1.1) will define a unique Markov process with continuous trajectories only under the condition that the coefficients b_i and σ_{ij} do not increase too fast as $|x| \to \infty$, and under the condition that the quadratic form of the matrix $\|a_{ij}(s, x)\|$ is not degenerate.

The following conditions are assumed to be satisfied in the sequel.

1. All coefficients of the operator L are bounded and sufficiently smooth everywhere in E_n, including the point $x = 0$.

2. For some continuous function $m(x)$, which is positive when $x \neq 0$, and for all real λ_i the inequality

$$\sum_{i,j=1}^{n} a_{ij}\lambda_i\lambda_j \geqslant m(x) \sum_{i=1}^{n} \lambda_i^2$$

is valid.

From the condition 1 and (1.3) it follows that

$$a_{ij}(x) = O(|x|^2), \qquad b_i(x) = O(|x|) \qquad (x \to 0) \qquad (1.4)$$

Under these hypotheses the equation (1.1) (or the operator L) defines a unique Markov process $X^{(m)}$ in the region $U_m = \{|x| > 1/m\}$ up to the instant τ_m when the boundary Γ_m of this region is reached.

In what follows, we shall make use of some concepts of the theory of Markov processes, such as the strong Markov process, a part of a process, and others. For the definition of these concepts see [3].

Lemma 1. Suppose that the conditions 1 and 2 are satisfied. Then there exists a unique strong Markov process in E_n with continuous trajectories, which does not vanish, and the part of which in U_m coincides with $X^{(m)}$.

Proof. Let X be some process satisfying the conditions of the lemma. Let us denote by τ_0 the instant when the trajectory of this process reaches the point $x = 0$. We shall prove first that for $x \neq 0$

$$P_x\{\tau_0 < \infty\} = 0 \qquad (1.5)$$

It is clear that for $|x| > 1/m$

$$P_x\{\tau_0 < t\} \leqslant P_x\{\tau_m < t\} \tag{1.6}$$

As is known (e.g. [4]), the function $v_m(t, x) = P_x\{\tau_m < t\}$ is the unique solution of the problem

$$\frac{\partial v}{\partial t} = L(x)\, v \tag{1.7}$$

satisfying the conditions $v_m(t, x)|_{|x|=1/m} = 1$, $v_m(0, x) = 0$. Let us now consider the function $w(t, x) = (t + 1)r^{-\alpha}$; the number α will be chosen later. One can easily verify that

$$\left(L - \frac{\partial}{\partial t}\right) w = -r^{-\alpha} + \alpha \Big[(\alpha + 2)\, r^{-\alpha-4} \sum_{i,j=1}^{n} a_{ij} x_i x_j - \\ - r^{-\alpha-2} \sum_{i=1}^{n} a_{ii}(x) - r^{-\alpha-2} \sum_{i=1}^{n} b_i(x)\, x_i\Big]$$

From (1.4) it follows that

$$\left(L - \frac{\partial}{\partial t}\right) w = -r^{-\alpha} + \alpha o(r^{-\alpha}) < 0$$

if α is sufficiently small. For such a choice of α, the function

$$w_m(t, x) = m^{-\alpha} w(t, x) - v_m(t, x)$$

satisfies, obviously, the conditions

$$\left(L - \frac{\partial}{\partial t}\right) w_m \leqslant 0, \quad w_m(0, x) \geqslant 0, \quad w_m(t, x)|_{|x|=1/m} \geqslant 0$$

From the principle of a maximum it follows that

$$w_m(t, x) \geqslant 0 \quad \text{for } t \geqslant 0,\ r = |x| \geqslant 1/m$$

Therefore

$$P_x\{\tau_m < t\} \leqslant (t + 1)\, r^{-\alpha} m^{-\alpha}$$

This, together with (1.6), implies (1.5).

We note that we have proved simultaneously also the uniqueness of the bounded solution of Cauchy's problem for the equation (1.7) in the region $E_n \times \{t > 0\}$.

Indeed, if u_1 and u_2 are two different solutions, and if $|u_i| < M$, then in a manner analogous to the above procedure we obtain

$$|u_1 - u_2| < (t + 1)\, 2Mr^{-\alpha}m^{-\alpha}$$

in the region $U_m \times \{t > 0\}$.

Taking the limit as $m \to \infty$, we obtain $u_1 \equiv u_2$. The uniqueness of the solution of the mixed problem is established in an analogous way.

Let us next prove that for any process satisfying the conditions of the lemma

$$P_0\{X(t) \equiv 0\} = 1 \tag{1.8}$$

Let A be the set $|x| = r \leqslant r_0$. We denote by τ_a the instant of the first encounter of the trajectory X of the process with the boundary of the set A, and by τ_m, as before, the instant of the first encounter of the trajectory of the process with the boundary $\Gamma_m = \{r = 1/m\}$. Because the process X is a strong Markov process, we obtain

$$P_0\{\tau_a < t\} = \int\limits_{y \in \Gamma_m} \int\limits_{u=0}^{t} P_0\{\tau_m \in du,\ X(\tau_m) \in dy\} P_y\{\tau_a < t - u\} \qquad (1/m < r_0) \tag{1.9}$$

The function $v(t, x) = P_x\{\tau_a < t\}$ satisfies the condition (1.6) in the region $D(\{r < r_0\} \times \{t > 0\})$, and the conditions $v(0, x) = 0$ and $v(t, x)|_{r=r_0} = 1$. (As was mentioned above, such a solution is unique.) After this it is not difficult to prove, with the aid of the maximum principle, that

$$P_x\{\tau_\alpha < t\} \leqslant \frac{r^\alpha}{r_0^\alpha}(t + 1) \quad \text{when } (t, x) \in D$$

if α is a sufficiently small positive number. Taking into account (1.9), we now obtain

$$P_0\{\tau_\alpha < t\} \leqslant (t + 1)\, r_0^{-\alpha} m^{-\alpha}$$

Taking the limit as $m \to \infty$, we obtain $P_0\{\tau_a < t\} = 0$ when $t > 0$, $r_0 > 0$. This implies (1.8). It is obvious that (1.5) and (1.8) imply the truth of the lemma.

In what follows, we shall assume that the conditions of the lemma are satisfied.

2. Conditions for stability. *Theorem 2.1.* For the stability of the trajectory $X(t) \equiv 0$ of the process X it is necessary and sufficient that in some neighborhood of the point $x = 0$ there exists a continuous non-negative function $V(x)$, which vanishes only at $x = 0$, and for which $LV(x) \leqslant 0$.

Proof. Let the positive number ε be given. We denote a solution of the problem by $U_m^{(\varepsilon)}(x)$

$$Lu = 0 \text{ when } 1/m < r < \varepsilon$$

$$u_m^{(\varepsilon)}(x)|_{r=1/m} = 0, \qquad u_m^{(\varepsilon)}(x)|_{r=\varepsilon} = 1$$

It is known [4], that

$$u_m^{(\varepsilon)}(x) = P_x\{\sup|X(t)| > \varepsilon\ (0 \leqslant t \leqslant \tau_n)\}$$

Taking into account (1.5), we obtain easily

$$u_\varepsilon(x) = P_x\{\sup|X(t)| > \varepsilon\ (0 \leqslant t < \infty)\} = \\ = \lim_{m\to\infty} P_x\{\sup|X(t)| > \varepsilon\ (0 \leqslant t \leqslant \tau_m)\} \tag{2.1}$$

From (2.1) it follows that the function $u_\varepsilon(x)$ too satisfies the equation $Lu = 0$ (as the limit of a monotone sequence of "harmonic" functions). From this and from (1.2) there follows at once the necessity of the conditions of the theorem because one can take the function $u_\varepsilon(x)$ itself for the function V. (From the stronger principle of a maximum it follows that $u_\varepsilon(x) > 0$ everywhere except at the point $x = 0$.)

It is not difficult to establish the sufficiency of the conditions of the theorem. Indeed, from the maximum principle it follows that

$$u_m^{(\varepsilon)}(x) \leqslant \frac{V(x)}{\min_{|y|=\varepsilon} V(y)}$$

Taking the limit in this inequality first as $m \to \infty$, and then as $x \to 0$, we obtain (1.2). This proves the theorem.

Note. For a process which is not homogeneous in time but corresponds to the operator $L(t, x)$, one can prove Lemma 1, and an analog to Theorem 1 if the condition (1.3) is satisfied uniformly in t. We note here only the following sufficient condition of stability for this case.

Theorem 2.2. For the stability of the trajectory $X(t) \equiv 0$ of the process $X = \{X(t), P_{s,x}\}$ it is sufficient that there exist a positive definite (in the sense of [5]) function $V(s, x)$ for which $\partial V/\partial s + L(s, x)V \leqslant 0$.

Theorem 2.1 is well suited for establishing stability, but it is quite difficult to use it for determining instability in concrete processes. Therefore, we shall give another condition for instability which can be verified more easily.

Theorem 2.3. A necessary condition for the trajectory $X(t) \equiv 0$ of the process X being unstable is that in the neighborhood of $x = 0$ there exist a function $W(x)$ such that $W(x) \to \infty$ when $x \to 0$, and $LW \leqslant 0$ at any point of this neighborhood except at the point $x = 0$.

Proof. The function

$$u_m(x) = P_x\{\sup |X(t)| < \varepsilon \ (0 \leqslant t \leqslant \tau_m)\}$$

satisfies, for $1/m < |x| < \varepsilon$, the equation $Lu_m = 0$ and the conditions

$$u_m(x)|_{r=1/m} = 1, \qquad u_m(x)|_{r=\varepsilon} = 0$$

From the maximum principle it follows that

$$u_m(x) \leqslant \frac{W(x)}{\min_{|y|=1/m} W(y)}$$

Hence

$$P_x\{\sup |X(t)| < \varepsilon \ (0 \leqslant t < \infty)\} = \lim_{m\to\infty} P_x\{\sup |X(t)| < \varepsilon \ (0 \leqslant t \leqslant \tau_m)\} \leqslant$$
$$\leqslant W(x) \lim_{m\to\infty} \frac{1}{\min_{|y|=1/m} W(y)} = 0$$

It follows, therefore, that $P_x\{\sup|X(t)| > \varepsilon \ (0 \leqslant t < \infty)\} = 1$ when $x \neq 0$. This establishes the theorem.

Note. **From what has been proved it follows that under the conditions of the theorem the trajectory $X(t) \equiv 0$ is "uniformly unstable" in the sense that for every initial point $x \neq 0$ a particle will move away from a position of equilibrium with probability one.**

We shall consider still another definition of stability.

Definition. A trajectory $X(t) \equiv 0$ of the process $X = \{X(t), P_{s,x}\}$ is said to be asymptotically stable for $t \geqslant t_0$ if the condition (1.2) is satisfied, and if

$$\lim_{x\to 0} P_{s,x}\{\overline{\lim_{t\to\infty}} |X(t)| = 0\} = 1 \qquad \text{for } s \geqslant t_0$$

One can show that for the case of a process that is homogeneous in time, and which satisfies the conditions of Section 1, stability of the trajectory $X(t) \equiv 0$ implies asymptotic stability.

For nonhomogeneous processes this is not true in general, but one can give sufficient conditions for asymptotic stability also in this case.

3. Stability of a one-dimensional process. In this section it will be assumed that the process X is described by the operator

$$a(x)\frac{\partial^2}{\partial x^2}+b(x)\frac{\partial}{\partial x} \text{ в } E_1$$

where we restrict ourselves, in accordance with the assumptions of Section 1, to the case when

$$a(x) = a_0x^2 + o(x^2) \quad (a_0 \geqslant 0), \qquad b(x) = b_0x + o(|x|) \quad \text{as } x \to 0$$

(In this case it is not difficult to write out the necessary and sufficient conditions for stability. For this it is only necessary that the point $x = 0$ be an attractive unattainable point of the process when $x > 0$, and when $x < 0$ [6].)

Theorem 3.1. The trajectory $X(t) \equiv 0$ of the process X is stable (asymptotically) when $b_0 < a_0$, it is unstable when $b_0 > a_0$.

Proof. 1) Suppose tnat $b_0 < a_0$. We consider the function $V(x) = |x|^\gamma$, where γ is some positive number less than $1 - b_0/a_0$.

Obviously

$$LV = a_0x^2\gamma(\gamma-1)|x|^{\gamma-2} + b_0|x|\gamma|x|^{\gamma-1} + o(|x|^\gamma) =$$
$$= \gamma|x|^\gamma[a_0(\gamma-1)+b_0] + o(|x|^\gamma) < 0$$

in a sufficiently small neighborhood of the point $x = 0$. Hence, the function $V(x)$ satisfies the conditions of Theorem 2.1.

2) Suppose that $b_0 > a_0$. One can verify that the function $W(x) = -\ln|x|$ satisfies the conditions of Theorem 2.3, this proves the theorem.

We shall give some consequences of this theorem. Suppose e.g. the coefficient of diffusion $a(x) = o(x^2)$ (i.e. $a_0 = 0$).

Then Theorem 3.1 shows that the asymptotic stability based on the first approximation of the trajectory $x(t) = 0$ of the random process described by the equation

$$\partial x / dt = b(x) \tag{3.1}$$

will guarantee the stability of the random process X with the same coefficient $b(x)$ described by the equation

$$dx / dt = b(X) + \sqrt{a(x)}\,\dot{\xi} \tag{3.2}$$

In case of instability of the linear approximation for the process (3.1), the trajectory $X(t) \equiv 0$ for the process (3.2) will also be unstable. It is not difficult to give examples which show that in case of neutrality of the first approximation ($b_0 = 0$) for the process (3.1), one can have either stability or instability for the process (3.2).

It is interesting to note also that an unstable (even in the linear approximation) equilibrium of the process (3.1) will pass into a stable one if one imposes the "randomness" $\sqrt{[a(x)]}\,\dot{\xi}$ provided $a_0 > b_0$. The examples given in the next section show that this can happen, seemingly, only in the one-dimensional case.

4. Examples of investigations of stability in multidimensional processes.

1. Let the Markov process X correspond to the operator

$$L(x) \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i}$$

We shall assume that the system

$$dx_i / dt = b_i(x) \qquad (i = 1, \ldots, n) \tag{4.1}$$

is asymptotically stable in its linear approximation. Let us assume, furthermore, that

$$a_{ij}(x) = o(|x|^2) \qquad (x \to 0) \tag{4.2}$$

It is not difficult to show now that the process X is also stable (for the one-dimensional case this was done above). Indeed, it is known [7, p.62] that for the system (4.1) there exists a positive-definite quadratic form $V(x)$ for which the principal part of the expression

$$b_1(x)\,\partial V / \partial x_1 + \ldots + b_n(x)\,\partial V / \partial x_n$$

represents a negative-definite quadratic function $U(x)$. Hence, in view of (4.2), we have

$$LV = U(x) + o(|x|^2) < 0$$

for sufficiently small $|x|$. This permits us to apply Theorem 2.1.

2. Let the operator $L(x)$ $(x \to 0)$ have the form

$$L_1(x) \equiv \sum_{i=1}^{n} [b_i x_i + o(|x|)] \frac{\partial}{\partial x_i} + \left[\sum_{i,j=1}^{n} a_{ij} x_i x_j + o(|x|^2)\right] \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \qquad (4.3)$$

where all the numbers $b_i < 0$. It is then obvious that the system "without randomness" is stable (asymptotically) in its linear approximation.

Let us consider the auxiliary function $V = (x_1^2 + \ldots + x_n^2)^\alpha$ (the number α will be chosen later). Obviously

$$L_1 V = 2\alpha (x_1^2 + \ldots + x_n^2)^{\alpha-1} \left\{ \sum_{i=1}^{n} b_i x_i^2 + \right.$$
$$\left. + \left[\sum_{i,j=1}^{n} a_{ij} x_i x_j\right](n - 2 + 2\alpha) + o(|x|^2) \right\}$$

If $n \leqslant 2$, then it is clear that if we take $\alpha > 0$ sufficiently small we can make sure that $L_1 V < 0$ in some neighborhood of the point $x = 0$ for arbitrary numbers a_{ij}. If, however, $n > 2$, then it is not difficult to choose the numbers a_{ij} so that $L_1 V < 0$ for some $\alpha < 0$. Making use now of Theorems 2.1 and 2.3, we obtain the following conclusion.

In case $n = 2$, and $b_i < 0$, the trajectory $X(t) \equiv 0$ is stable for the process which corresponds to the operator (4.3) for arbitrary values of the coefficients a_{ij}. However, in case $n > 2$, this process is stable if the a_{ij} are sufficiently small, and it is unstable if these coefficients are sufficiently large.

3. Let us consider the system

$$\frac{dX_1}{dt} = X_2 + \sigma(X_1, X_2)\dot{\xi}_1, \qquad \frac{dX_2}{dt} = -X_1 + \sigma(X_1, X_2)\dot{\xi}_2$$

It is clear that the position of equilibrium of this system in the absence of random disturbances ($\sigma(x_1, x_2) \equiv 0$) is stable, but not asymptotically.

Let us introduce the notation

$$L_2(x) \equiv x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + \sigma^2(x_1, x_2)\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$$

It is clear that $L_2 W = 0$ if $W = -\ln(x_1^2 + x_2^2)$ and, hence $W(x)$ satisfies the conditions of Theorem 2.3 provided $\sigma(x) \neq 0$ when $x \neq 0$. Therefore, the trajectory $X(t) \equiv 0$ is unstable for this process.

The example shows that a system "without randomness", i.e. non-asymptotically stable, can pass into an unstable system if one adds diffusion terms of arbitrarily high order of smallness when $x \to 0$.

In conclusion, let us consider some problems.

(a) The following cases may be of interest, when the operator $L(x)$ degenerates not at individual points but on a manifold of dimensions less than n. This gives rise to the problem on the stability of the set of trajectories contained in this manifold. It is not difficult to see that the theorems of Secion 2 can be modified so that they will apply to this case. We note also that it is exactly such a case (under a different definition of stability, and under different assumptions) that was treated in the interesting work [1], which stimulated the writing of the present note.

(b) It was shown above that (4.2) and the asymptotic stability of the linear approximation to the system (4.1), guarantee the stability of the process X. One can, probably, prove that (4.2) and the instability of the linear approximation of (4.1) imply the instability of the process X. (This has been proved for a particular case in Section 3). It seems that for the proof of this assertion it would be necessary to obtain an effective sufficiency condition for instability, which could be applied to a more general class of cases than Theorem 2.3.

(c) It is clear from the examples presented that in many cases the solution of the problem on the stability of a process X corresponding to the operator

$$L(x) = \sum_{i,\,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}$$

can be reduced to the same problem for the process of the "first approximation" X°, described by the operator

$$L^{\circ}(x) = \sum_{i,\,j=1}^{n} a_{ij}^{\circ}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^{\circ}(x)\frac{\partial}{\partial^{r}}$$

where the $a_{ij}^{\circ}(x)$ are quadratic, while the $b_i^{\circ}(x)$ are linear forms that are the first non-vanishing terms of the expansion of the coefficients $a_{ij}(x)$ and $b_i(x)$ by Taylor's formula in the neighborhood of the point $x = 0$. It would be of interest to obtain fairly general criteria of stability of the process X°.

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